

# Particle Path Formulation of Quantum Mechanics

by

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## Abstract

An extension of the classical action principle obtained in the framework of the gauge transformations, is used to describe the motion of a particle. This extension assigns many, but not all, paths to a particle. Properties of the particle paths are shown to impart wave like behaviour to a particle in motion and to imply various other assumptions and conjectures attributed to the formalism of Quantum Mechanics. The Klein-Gordon and other similar equations are derived by incorporating these properties in the path-integral formalism.

## 1 Introduction

This paper describes a recent approach to mechanics based on an extension of Hamilton's action principle, obtained by a process of completion in the framework of the gauge transformations.

In Sec. 2, a motivation for the extension is developed by examining the action principle and by reformulating it in terms of the gauge transformations. In Sec. 3, the extension termed the gauge mechanical principle, is formulated, interpreted and its solutions are classified. In Sec. 4 the solutions of the gauge mechanical principle are used to describe the motion of a free particle, the behaviour of the particles in a double slit experiment and the Aharonov-Bohm effect. Although the present formulation excludes some trajectories from the collection of physical paths, the results are sufficient to justify Feynman's path integral formalism to formulate mechanics, at least approximately (Sec. 5). The properties of the physical paths are therefore incorporated in the path-integral formalism in Sec. 5, to derive a generalized Schrödinger type equation which is then reduced to a set of infinitely many four-dimensional equations, one of them being the Klein-Gordon equation. In Sec. 6, some additional results are

quoted and directions for further development are indicated. In conclusion in Sec. 7, summary of the results is used to justify the present formulation of mechanics.

This approach to mechanics was developed independently of any direct considerations of the behaviour of particles in experimental settings including the double slit experiment. However, its implications lead one to consider the following experimental observations and somewhat unorthodox conclusions that might be drawn from them.

In the double slit experiment, photons, electrons and other physical entities that are normally considered particles, demonstrate their particle nature if observed individually. However, if many are allowed to pass through the slits, together or one after the other, then an interference-like pattern of intensity emerges on the screen [1]. Since a wave would produce such a pattern, it is assumed that each particle also has a wave character. Quantum Mechanics accepts this duality by attaching a probability wave with a particle in motion i.e. the wave determines the probability of finding a particle in a certain space-time region. This fusion of wave and particle nature creates most of the logical difficulties with Quantum Mechanics [1,2]. The observation in the double slit experiment is viewed about the most puzzling mystery of nature. Also, its understanding is considered pivotal to the resolution of most of the paradoxical situations arising in the microscopic phenomena described by Quantum Mechanics.

While a wave would produce the intensity distribution of the type observed in the double slit experiment, the converse is not necessarily true i.e. the observation of this pattern does not prove that it was produced by a wave. A closer scrutiny of the experimental observations suggests an alternative possibility.

Conclusions based on relevant observations identify the observed entity as a particle when emitted or absorbed. Experiments designed to reveal its wave nature during travel observe each individual with particle like attributes. Therefore it may be possible to describe the experimental observations by associating a particle like trajectory with each of the entities. These observations on a number of particles suggest the possibility of the existence of a collection of paths out of which each particle takes one, probably randomly. This collection must be endowed with some characteristics which are responsible for the inclusion of more paths ending about the bright regions and exclusion of others. Therefore it appears more reasonable to build a theory of mechanics by characterizing the collection of particle paths rather than attempting to fuse mutually exclusive wave and particle behaviours. If this view is adopted, then the effect of an observation on its outcome must be the result of the disturbance suffered by the particle and hence, must be described in this manner. This philosophy has its origin in Fermat's principle of stationary time in light and Hamilton's principle of stationary action in classical mechanics. Both of these theories are geometrical in nature instead of mechanical, although Hamilton's principle is equivalent to Newton's second law which gives an impression of being a mechanical theory.

The implications of the present extension of Hamilton's action principle are in accordance with inferences that could be drawn from the experimental observations as discussed above. To be precise, the extension yields a collection of infinitely many, but not all, paths for a particle to follow which are endowed with some properties by virtue of the fact that they are the solutions of the extended principle. These properties are shown to describe the behaviour of particles in a double slit experiment and in the Aharonov-Bohm experiment without invoking the usual assumption of probability waves or the formalism of Quantum Mechanics. The results are shown to justify Feynman's path integral formulation and used in this framework to derive a generalized Schrödinger type equation. Properties of the particle paths are used to reduce the general equation into infinitely many four dimensional equations, one of them being the

Klein-Gordon equation.

This formulation yields as results, the assumptions underlying the standard Quantum Mechanics and various other intuitive conjectures usually attributed to the formalism of Quantum Mechanics. However there are some differences between the consequences of the present formulation and the standard Quantum Mechanics which are indicated in the sequel.

This paper is more detailed than the paper to appear in the conference proceedings.

## 2 The Action Principle

In this section, a reformulation of the action principle is presented that is well suited for its extension in the framework of the gauge transformations.

Let  $L(\dot{x}, x, \tau)$  be a Lagrangian defined on curves in a manifold  $\mathcal{M}$ . While the action principle may be formulated in any differentiable manifold, for the present we shall have occasion only to deal with the Minkowski space. For a path  $\rho(AB) = x(\tau)$  with  $x(\tau_1) = A$ ,  $x(\tau_2) = B$ , the action functional  $S_{BA}(\rho) = S(\tau_1, \tau_2)$  is given by

$$S(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} L(\dot{x}, x, \tau) d\tau \quad (1)$$

The action principle characterizes the particle path(s) by requiring the action to be stationary i.e.

$$\delta S = S_{BA}(\rho') - S_{BA}(\rho) = 0 \quad (2)$$

up to the first order in  $\delta x$  where  $(x + \delta x)(\tau) = \rho'$ . The end points A and B are kept fixed and correspond to the same parameter values as the undistorted curve i.e.  $\tau_1$  and  $\tau_2$  respectively. Eq. (2) is expected to hold for all curves  $\rho'$  in a small neighbourhood of the solution  $\rho$  if it exists.

Some conceptual clarity is gained in describing the action principle by considering the analogue of  $x(\tau)$  in  $\mathcal{M}'$  obtained from  $\mathcal{M}$  by including  $\tau$  as an additional coordinate [3, Ch. 1.1]. Thus the curve  $x(\tau)$  in  $\mathcal{M}$  corresponds to the set of points  $(x(\tau), \tau)$  in  $\mathcal{M}'$ , eliminating a need for an explicit reference to the parameterization.

Eq. (2) in  $\mathcal{M}'$  takes the following form:

$$S_{ABA}(\rho_c) = 0 \quad (3)$$

up to the first order in  $d\sigma$  where  $\rho_c$  is the closed curve in  $\mathcal{M}'$  obtained as the union of  $\rho'$  and  $\rho$  inverse i.e.  $\rho_c$  traces the path  $\rho'$  from  $(A, \tau_1)$  to  $(B, \tau_2)$  and then inverted  $\rho$  from  $(B, \tau_2)$  to  $(A, \tau_1)$ , and  $d\sigma$  is the area enclosed by  $\rho_c$ . Eq. (2), equivalently (3) yields the Euler-Lagrange equation that describes the particle path.

Consider a charged particle in an electro-magnetic field which may be described by the Lagrangian  $L = L^P - \phi_\mu \dot{x}^\mu$ , where  $L^P = \frac{1}{2}m(\dot{x}_\mu \dot{x}^\mu + 1)$ , and  $\phi_\mu$  are the electro-magnetic potentials. Quite frequently, a homogeneous Lagrangian is used instead, but  $L$  is more convenient. Both formulations are equivalent with  $\tau$  being the proper time. The Lorentz equation describing the path of a charged particle in an electro-magnetic field is the solution of (2) or (3) with the Lagrangian given by  $L$ , i.e. the particle path is characterized by

$$\oint L^P d\tau - \oint \phi_\mu dx^\mu = 0 \quad (4)$$

up to the first order.

Eq. (4) relates this characterization of the particle-path with the gauge transformations as follows. Weyl introduced the notion of the gauge transformations by proposing that a rigid measuring rod must be gauged at every space-time point according to the rule [4]

$$d\Phi_A = \alpha\phi_\mu dx^\mu \Phi_A \quad (5)$$

where  $d\Phi_A$  is the change suffered by a rod of length  $\Phi_A$  at  $A$  under the infinitesimal displacement  $dx$  and  $\alpha$  is a constant. From (5), the length  $\Phi_{BA}$  at  $B$  of the same rod transported along  $\rho$  is given by

$$\Phi_{BA}(\rho) = U_{BA}(\rho)\Phi_A \quad (6)$$

where

$$U_{BA}(\rho) = \text{Exp} \left( \alpha \int_{\rho(AB)} \phi_\mu(x) dx^\mu \right) \quad (7)$$

is the group element associated with  $\rho(AB)$ . The Lie algebra element associated with the displacement  $dx$  is  $\alpha\phi_\mu dx^\mu$ . It is clear that the action principle describes the particle-path in terms of the gauge Lie algebra element, associated with the curves of the type  $\rho_c$ .

Eq. (3) may be expressed as

$$\exp(\alpha S_{ABA}(\rho_c)) = 1 \quad (8)$$

equivalently as

$$U^P_{ABA}(\rho_c) = U_{ABA}(\rho_c) \quad (9)$$

up to the first order in  $d\sigma$ , with  $U^P_{BA}(\rho) = \exp(\alpha S^P_{BA}(\rho))$  where  $S^P_{BA}(\rho)$  is the free particle part of the action associated with  $\rho(AB)$ .

### 3 The Gauge Mechanical Principle

The formulation of the action principle described in Sec. 2 indicates that the classical description of motion is deficient in gauge group theoretical terms. This description limits itself to a characterization of particle-path(s) in terms of the Lie algebra elements, equivalently, the infinitesimal gauge group elements, which is accurate only up to the first order. This characterization is local in nature. Additional information that may be available in the global group elements is not utilized in the action principle. Therefore a description in terms of the group elements should be expected to be more complete. This deficiency can easily be corrected by including the higher order terms in addition to other adjustments if need be. Any such characterization must reduce to

$$\exp(\alpha S_{ABA}(\rho_c)) = 1 \quad (10)$$

Eq. (10), although an extension, limits itself to considering only the closed curves in  $\mathcal{M}'$  while the group elements are defined for all curves. If the action principle is to be extended in terms of the gauge group elements, then this restriction becomes redundant. To achieve appropriate generality consistent with the domain of definition of the gauge group elements, the action principle should be extended to

$$\kappa^{-1}(B)\exp(\alpha S_{BA}(\rho))\kappa(A) = 1 \quad (11)$$

where  $\kappa$  is as yet an undetermined function which cancels out for the closed curves. The characterization of particle paths by (11) has been termed the gauge mechanical principle [5]. Its solutions will be called the physical paths which a particle is allowed to follow.

It should be remarked that there is no logical deficiency or inconsistency in the action principle itself. The argument here is that the action principle provides an incomplete description of motion in gauge group theoretical terms. Prejudice in favour of the group elements in comparison with the Lie algebra elements, in favour of the global in comparison with the local, is a matter of metaphysical conviction.

In the above, we have provided arguments to justify the present extension of the classical action principle, not a derivation of the gauge mechanical principle. These arguments are to some extent irrelevant as far as the matter of the extension is concerned. The fact that (11) reduces to (8) with appropriate restrictions is sufficient to prove that (11) is an extension of the action principle.

Furthermore, the gauge mechanical principle by itself may be made the basis of a formulation of mechanics whether it is an extension of the action principle or not. All that is required is that it provide an adequate description of the motion of particles. The fact that it is an extension of the action principle serves only to relate the resulting mechanics with Classical Mechanics. In the remainder of this section we clarify the principle further and present its alternative statements.

First we relate the gauge mechanical principle with Newton's second law of motion. Some such relation should be expected as the action principle is equivalent to Newton's law. We limit here to the motion of a charged particle in an electro-magnetic field which illustrates the relation without cluttering the concepts with unnecessary generalities.

Eq. (11) may be expressed as

$$U^P_{BA}(\rho) = \kappa(B)U_{BA}(\rho)\kappa^{-1}(A) \quad (12)$$

The right side of (12) is equal to  $(1 + \alpha F_{\mu\nu} d\sigma^{\mu\nu})$  for infinitesimal closed curves, where  $F_{\mu\nu}$  are the components of the field tensor. The left side under the same conditions reduces to  $(1 - (p_\mu \dot{p}_\nu - \dot{p}_\mu p_\nu) d\sigma^{\mu\nu})$  where  $p_\mu$  are the components of the canonical momentum. This equality is equivalent to the Lorentz equation, equivalently, Newton's second law [6].

The gauge mechanical principle may also be interpreted in terms of Weyl's original notion of gauging a rigid measuring rod as follows. Recall that  $\Phi_{BA}$  is the length of Weyl's rod at B transported along  $\rho(AB)$  while its length at A was  $\Phi_A$ . Eq. (12) may be expressed as

$$\Phi^P_{BA}(\rho) = \kappa(B)\Phi_{BA}(\rho) \quad (13)$$

where  $\Phi^P_{BA}(\rho) = U^P_{BA}(\rho)\Phi^P_A$ , and  $\Phi^P_A = \kappa(A)\Phi_A$ . Weyl's gauge transformations determine the effect of a field on the rigid measuring rod. One may take another rod of length  $\Phi^P_A$  and transport it along a given curve  $\rho(AB)$ . Let  $\Phi^P_{BA}$  be its length at B determined as above without any reference to the field. The gauge mechanical principle requires that Weyl's gauge and the present gauge must return essentially in the same relation as they began with at A for  $\rho(AB)$  to be a particle path.

In Newton's second law, one equates a force-like quantity determined solely by the curves in the space-time manifold with the force postulated by an independent law. In the present formulation, one computes the change in the length of the measuring rod solely from the curves in  $\mathcal{M}'$  without any reference to the field, which is then related to the change in Weyl's rod. It is not necessary to set  $\Phi^P_A = \Phi_A$  as it would limit generality without adequate justification. It is sufficient that a precise map between  $\Phi^P_A$ ,  $\Phi^P_{BA}$  and  $\Phi_A$ ,  $\Phi_{BA}$  be available. This is consistent with (11) and (12), as the equality (11) for closed curves implies only the group equivalence (12) for general curves.

The function  $\kappa$  in the above appears as a requirement of the mathematical generality as there is no justification for imposing further restrictions on (12). However, for a physical theory,  $\kappa$  must have a clearer physical significance which we discuss below.

The elements  $U$  and  $U^P$  appearing in (12) pertain to the interiors of the respective curves. As such there is no consideration of the initial physical state of a particle or of local interventions at B or elsewhere. Obviously the physical paths for two particles in different physical states should be expected to be different in the same field. Therefore it is legitimate to interpret  $\kappa$  as representing the physical state of the particle. Interaction with the detecting instrument is local in nature and has a direct impact on the physical state of the particle. Therefore such interactions are also included in  $\kappa$  by way of the physical state of the particle. A precise computation of  $\kappa$  is not necessary for a variety of experimental situations. For example, in the double slit experiment, the particles passing through two slits at A and A' are prepared by the same physical process and are identical in every other respect. Therefore, it is legitimate to conclude that particles at A and A' are in the same physical state even if it may not be precisely defined. Hence, we may set  $\kappa(A) = \kappa(A')$ . Similarly, two beams meeting at B interact with the same instrument. Therefore B is not only geometrically the same point for two paths  $\rho(AB)$  and  $\rho(A'B)$ , it is also physically equivalent. Therefore  $\kappa$  has the same value for two beams at B. This will be found sufficient for the description of the behaviour of particles in the double slit experiment. The same comment applies to various other experimental situations.

Consider a free particle travelling from A to B along  $\rho(AB)$  without any interactions including the intrusion of a detecting instrument. In this situation, the physical state of the particle must remain unchanged. Therefore we shall assume that for a free particle  $\kappa(A) = \kappa(B)$  for all points A and B i.e.  $\kappa$  is constant. This extends Newton's first law. The effects of interactions on  $\kappa$  may also be computed. A detailed description of such computations is beyond the scope of this article but it is indicated below to an extent necessary for the clarity of the gauge mechanical principle.

Since for a free particle  $\kappa$  is constant, any change in its value must be a result of an interaction. In standard interventions, a precise value of the interaction is unknown e.g. a detecting instrument but the instantaneous change in the classical momentum may be computed or estimated with sufficient accuracy. This information is sufficient to compute the change  $\Delta S$  in the action caused by the interaction. The change in the value of  $\kappa$  is then given by  $\exp(\alpha \Delta S_{BA}(\rho))$ .

The value of  $\alpha$  still remains undetermined which we obtain below. For a free particle, the physical paths are defined by

$$\exp(\alpha S_{BA}(\rho)) = 1 \quad (14)$$

In general, the action  $S_{BA}(\rho)$  is real and non-zero. There are configurations of curves with total action equal to zero but there is no justification for restricting the description of motion to such curves only. Therefore  $\alpha$  must be purely imaginary which may be set equal to  $i$  in appropriate units.

The representations of the gauge mechanical principle given by (11), (12) and (13) are essentially equivalent. Reference to the gauging of the measuring rod is inconsequential for the following developments. Reference to one of the representations, therefore, will include others as well.

Solutions of (13) are identified by their equivalence classes as follows. Let  $V_{BA}(\rho) = \kappa(B)V'_{BA}(\rho)\kappa^{-1}(A)$  with  $V'_{BA}(\rho) = U_{AB}^P(\rho)U_{BA}(\rho)$ , and let  $\{B_j\}$  be a set of points on  $\rho(AB)$  such that  $V_{B_j A}(\rho)\Phi_A^P = \Phi_A^P$ . If one member of  $\{B_j\}$  is a physical point with respect to  $\{\rho(AB), \Phi_A^P\}$ , then this is also the case for each  $j$ . Thus the equivalence class  $\{B_j\}$  so defined characterizes

the solutions  $\{\rho(AB_j)\}$ . A natural order is defined on  $\{B_j\}$  by setting  $B_j$  to be the  $j$ th closest member to  $A$ . Let  $\{B_j^k\}$ ,  $k = 1, 2, \dots$ , be such ordered equivalence classes with respect to  $\{\rho(AB^k), \Phi_A^P\}$ . The set  $\zeta_j = \{B_j^k\}$  defines a physical ‘surface’ for each  $j$ .

For a free particle, the physical paths are the solutions of (14) which reduces to

$$\exp\left(im \int_{\rho(AB)} u_\mu dx^\mu\right) \Phi_A^P = \Phi_A^P \quad (15)$$

The equivalent points  $\{B_j\}$  on these curves satisfy

$$m \int_{\rho(B_j, B_{j+1})} u_\mu dx^\mu = 2\pi$$

Along the paths characterized by a constant velocity  $\bar{u}$ ,  $B_j$  and  $B_{j+1}$  are thus separated by the de Broglie wavelength  $2\pi/m\bar{u}$  and the length of a physical path is its integral multiple.

Consider a source-detector system with source at  $A$  and detector at  $B$ . A curve  $\rho(AB)$  will be called monotonic if the parameter value increases or decreases monotonically along the curve. By convention,  $\tau$  will be assumed to increase from  $A$  to  $B$ . A particle starting at  $A$  and confined to  $\rho(AB)$  is observable at  $B$  if and only if  $\rho(AB)$  is physical. If  $\theta$  is the intensity associated with  $\rho(AB)$  at  $A$  then the intensity transmitted to  $B$  by this path must be equal to  $\theta$ .

A union of physical paths is obviously physical. Also a union of non-physical monotonic curves can be physical. For example, let  $\rho(AB)$  be a monotonic physical path with the associated physical points  $\{B_j\}$  and let  $C$  be a point in the interior of  $\rho(B_j B_{j+1})$ . Then the union of  $\rho(B_j C)$  and  $\rho(C B_{j+1})$  is  $\rho(B_j B_{j+1})$  and the union of  $\rho(B_{j-1} C)$  and  $\rho(C B_j)$  is  $\rho(B_{j-1} B_j)$ , both of which are physical. However, these trivial constructions are redundant as they are indistinguishable from the paths of the type  $\rho(B_k B_{k+l})$ . A significant, non-trivial class of such paths is described below.

Consider a configuration of two curves  $\rho(AB)$  and  $\rho'(AB)$  with  $\rho_c(ABA)$  being the union of  $\rho'(AB)$  and  $\rho(BA)$ . According to the present prescription, if (13) is satisfied then this is a physical configuration. Since the evolution parameter increases from  $A$  to  $B$  along both of the curves, particle must travel from  $A$  to  $B$  along  $\rho(AB)$  and  $\rho'(AB)$ . Therefore  $\rho(AB)$  and  $\rho'(AB)$  offer equally likely alternatives for the transmission of a particle from  $A$  to  $B$ , even if  $\rho(AB)$  and  $\rho'(AB)$  may not be physical. The case of the alternatives of the type  $\rho(AB)$  and  $\rho'(CB)$  is treated similarly. To be precise, let the parameter value at  $B$  be  $\tau_B$ . According to the above convention,  $\tau$  increases from  $C$  to  $B$  along  $\rho'(CB)$  and decreases from  $B$  to  $A$  along  $\rho(BA)$ . If  $V_{CBA}(\rho'')\Phi_A^P = \Phi_A^P$ , where  $\rho''$  is the union of  $\rho'(CB)$  and  $\rho(BA)$ , then  $\rho(AB)$  and  $\rho'(CB)$  offer likely alternatives. Such configurations of trajectories are referred to as the interfering alternatives. The intensity of particles transmitted to  $B$  by the equally likely alternatives must be equal to the sum of the intensities at  $A$  and  $C$  associated with the respective trajectories. Such a union of paths is indistinguishable from a pair of monotonic physical paths since  $\kappa(B)$  may be adjusted such that  $\rho'(CB)$  and  $\rho(AB)$  are both physical which does not alter the relevant physical content.

## 4 Physical Paths

As a prelude to a more precise treatment of motion in Sec. 5, an approximate description of a few phenomena is given in this section, which also clarifies the properties of a multiplicity of physical trajectories.

### 4.1 Motion of a particle.

Consider a physical system described by a Lagrangian  $L(\dot{x}, x)$  with  $\rho_s$  being the resulting classical path. For convenience, it is assumed that  $L$  does not depend on  $\tau$  explicitly. However,  $\tau$ -dependence may be included without a significant change in the following analysis. For a free particle,  $L = L^P$ . For an undisturbed particle, the equivalent points on  $\rho_s$  are given by

$$S(B_j, B_{j+1}, \tau_j, \tau_{j+1}) = 2\pi$$

where  $S(\ )$  denotes Hamilton's principal function. The action  $S_{B'A'}(\rho')$  along a trajectory  $\rho'(A'B')$  in a small neighbourhood of  $\rho(AB)$  is given by

$$\begin{aligned} (S_{B'A'}(\rho') - S_{BA}(\rho)) &= \int_{\rho(AB)} \delta x^\mu \left[ \frac{\partial L}{\partial x^\mu} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^\mu} \right] d\tau \\ &+ \left[ \frac{\partial L}{\partial \dot{x}^\mu} \delta' x^\mu - H \delta' \tau \right]_A^B + O(\delta^2) \end{aligned} \quad (16)$$

by standard methods. Here  $\delta' x^\mu$ ,  $\delta' \tau$  correspond to the variation of the end points  $A$ ,  $B$  to  $A'$ ,  $B'$ , and  $H$  is the Hamiltonian. The term  $O(\delta^2)$  is the integral along  $\rho(AB)$  of an argument, containing functions of second or higher order in  $(\delta x)$  and  $(\delta \dot{x})$ .

If  $\rho = \rho_s$ , then the first term on the right side of (16) is equal to zero. Hence  $S_{B'A'}(\rho') = S_{BA}(\rho_s)$  for some values of  $\delta' x = O(\delta^2)$ . Therefore the trajectories in a  $\delta x$  neighbourhood of a physical classical path  $\rho_s(B_j B_{j+k})$  are also physical and their end points are confined to  $(\delta^2)$  neighbourhoods of  $B_j$  and  $B_{j+k}$ . Thus the intensity transmitted by paths in a  $\delta x$  neighbourhood of a classical trajectory is concentrated in  $(\delta^2)$  neighbourhoods of the equivalent points on  $\rho_s$ . Let  $\rho$  be a path transmitting intensity outside  $(\delta^2)$  neighbourhood of  $\{B_j\}$ . Since  $\rho$  is not a solution of the Euler-Lagrange equation, the first term in (16) dominates which is  $O(\delta x)$ . Repeating the above argument, we have that the intensity transmitted by trajectories in a  $\delta x$  neighbourhood of  $\rho$  is spread over a  $\delta x$  neighbourhood of points outside  $(\delta^2)$  neighbourhood of  $\{B_j\}$ . Further, the magnitude of the first term in (16) increases as  $\rho$  is removed farther from the classical trajectory. Therefore the contribution to the intensity decreases accordingly. Some intensity is also transmitted by the interfering alternatives whose monotonic segments are non-physical. In a homogeneous space, such paths are roughly evenly distributed about the classical trajectory implying a uniform distribution of the associated intensity. The properties of such paths will be described in more detail in Sec. 4.2 where their impact is greater.

Assuming that the particles originate in a small region about a point  $A$ , intensity should be expected to be higher near the points equivalent to  $A$  and to decrease away from them, creating a wave-like pattern over a uniform background. On a classical scale, the segments between  $B_j$  and  $B_{j+1}$  are negligibly small. Also for macroscopic trajectories, the contribution of the first term in (16) is enormous as one moves away from a purely classical trajectory, owing to the large interval of integration. Therefore, the contribution to the variation of the intensity over a wavelength, between  $B_j$  and  $B_{j+1}$ , must come from extremely small neighbourhoods of the



long trajectories, and from larger neighbourhoods of the shorter ones, which are still small on a classical scale. Thus on a macroscopic scale, the particles from  $A$  to  $B$  travel along narrow beams centered about the classical trajectories.

## 4.2 The double-slit experiment.

The interfering alternatives play a prominent role in the double slit experiment. In this setup, identical particles are allowed to pass through two slits at  $A$  and  $A'$ , and collected on a distant screen at a point  $B$ . The following treatment is valid in the presence of a field. As explained in Sec. 4.1, the particle paths may be assumed concentrated about the classical trajectories from  $A$  to  $B$  and from  $A'$  to  $B$ . If one of the beams is blocked, then the intensity observed in a neighbourhood of  $B$  should behave as deduced in Sec. 4.1 for a free particle. However, if the intensity is transmitted by both of the beams, then a multitude of the interfering alternatives is allowed. Existence of such paths and their influence on the intensity distribution is studied next.

In view of the physical equivalence of  $A$  and  $A'$  and that of the particles, one has that  $\kappa(A) = \kappa(A')$ ,  $\Phi_A^P = \Phi_{A'}^P$ , and hence  $\Phi_A = \Phi_{A'}$ . However, because of an interaction with the detecting instrument at  $B$ ,  $\kappa(B)$  may not be equal to  $\kappa(A)$ . For the interfering alternatives, the value of  $\kappa(B)$  is the same for both of the monotonic segments (Sec. 3). Substitutions in (13) show that the paths are the solutions of

$$\exp \left[ i \left( \int_{\rho(AB)} dS(x, \tau) - \int_{\rho'(A'B)} dS(x, \tau) \right) \right] \Phi_A = \Phi_{A'} \quad (17)$$

For the classical trajectories  $\rho = \rho_s$  and  $\rho' = \rho'_s$ , (17) is solved by

$$(S_{BA}(\rho_s) - S_{BA'}(\rho'_s)) = 2\pi j$$

where  $j$  is an arbitrary integer and the action in this case is Hamilton's principal function or the arc-length in  $\mathcal{M}$ . Classical paths are characterized by a constant velocity  $\bar{u}$ . This reduces the solution to  $\Delta r = 2\pi j / m\bar{u}$ , where  $\Delta r$  is the difference between the path-lengths of  $\rho_s(AB)$  and  $\rho'_s(A'B)$ . Therefore  $\rho_s(AB)$  and  $\rho'_s(A'B)$  are interfering alternatives whenever  $\Delta r = 2\pi j / m\bar{u}$ .

Let  $B(\varepsilon)$  be the point on the screen such that

$$(S_{B(\varepsilon)A}(\rho_s) - S_{B(\varepsilon)A'}(\rho'_s)) = 2\pi(j + \varepsilon) \quad (18)$$

for a fixed  $j$  and each  $0 \leq \varepsilon \leq 1/2$ . In the following we study the variation of the intensity as  $\varepsilon$  varies in the prescribed interval which is sufficient to describe it on the entire screen.

It follows from the analysis of Sec. 4.1, that  $S_{CA}(\rho) = S_{B(\varepsilon)A}(\rho_s)$ ,  $S_{C'A'}(\rho') = S_{B(\varepsilon)A'}(\rho'_s)$ , for  $\rho, \rho'$  in  $\delta x$  neighbourhoods of  $\rho_s, \rho'_s$  respectively, where  $C$  and  $C'$  vary over a  $(\delta^2)$  neighbourhood of  $B(\varepsilon)$  on the screen for a fixed  $\varepsilon$ . Therefore, by varying the paths over a  $(\delta x)$  width of the beam and over a  $(\delta^2)$  neighbourhood of  $B(\varepsilon)$  it is possible to satisfy

$$(S_{DA}(\rho) - S_{DA'}(\rho')) = 2\pi(j + \varepsilon)$$

for most of the paths. In fact cancellations favour this equality which can be easily seen, in particular for the cases when  $\rho_s, \rho'_s$  are extremals as is presently the case. This conclusion is valid for other points in the vicinity of  $A$  and  $A'$  also. For  $\varepsilon = 0$ , this implies that there is a large concentration of interfering alternatives reaching about  $B(0)$  and hence the intensity in a  $(\delta^2)$

neighbourhood of  $B(0)$  is almost equal to the intensity in  $\delta x$  neighbourhoods of  $\rho_s(AB(0))$  and  $\rho'_s(A'B(0))$ . For  $\varepsilon \neq 0$ , the configuration of the paths  $\rho_s(AB(\varepsilon))$  and  $\rho'_s(A'B(\varepsilon))$  is obviously non-physical. From the above argument, a large number of paths in  $\delta x$  neighbourhoods of  $\rho_s(AB(\varepsilon))$  and  $\rho'_s(A'B(\varepsilon))$  are excluded from combining to form the interfering alternatives and hence unable to transmit the intensity in a  $(\delta^2)$  neighbourhood of  $B(\varepsilon)$ . Still there are many paths capable of transmitting intensity about  $B(\varepsilon)$  for  $\varepsilon \neq 0$ , which are described below.

It follows from (16) that for trajectories  $\rho(AB(\varepsilon))$ ,  $\rho'(A'B(\varepsilon))$  in  $\delta x$  neighbourhoods of  $\rho_s(AB(\varepsilon))$ ,  $\rho'_s(A'B(\varepsilon))$  respectively,

$$(S_{B(\varepsilon)A}(\rho) - S_{B(\varepsilon)A}(\rho_s)) = O(\delta^2)$$

and

$$(S_{B(\varepsilon)A'}(\rho') - S_{B(\varepsilon)A'}(\rho'_s)) = O(\delta^2)$$

We have used the fact that the first term on the right side of (16) is zero as the curves are varied about the classical trajectories and the second term is zero as the end points are kept fixed. For these curves, we have

$$(S_{B(\varepsilon)A}(\rho) - S_{B(\varepsilon)A'}(\rho')) = 2\pi(j + \varepsilon) + O(\delta^2) \quad (19)$$

Since there are distortions for which  $O(\delta^2)$  term is non-zero and its magnitude is large in natural units, it is possible to adjust the curves  $\rho$ ,  $\rho'$  such that

$$(S_{B(\varepsilon)A}(\rho) - S_{B(\varepsilon)A'}(\rho')) = 2\pi k \quad (20)$$

with  $k = j$  or  $(j + 1)$ , most likely  $j$ . This implies that  $\rho(AB(\varepsilon))$  and  $\rho'(A'B(\varepsilon))$  form a pair of interfering alternatives. Since  $\rho(AB(\varepsilon))$ ,  $\rho'(A'B(\varepsilon))$  are non-classical trajectories, it follows as in Sec. 4.1 that while there is a multitude of paths satisfying (20), in  $\delta x$  neighbourhoods of the central paths, their end points are spread over a  $\delta x(\varepsilon)$  neighbourhood of  $B(\varepsilon)$ . This implies that the amount of intensity that is concentrated in a  $(\delta^2)$  neighbourhood of  $B(0)$  is spread over a  $\delta x(\varepsilon)$  neighbourhood of  $B(\varepsilon)$ . Consequently, a rapid decrease in the intensity is expected as  $\varepsilon$  increases away from zero.

As  $\varepsilon$  increases further, it is seen from (19) that the neighbourhood  $\delta x$  must be increased to satisfy (20), i.e.  $\rho(AB(\varepsilon))$ ,  $\rho'(A'B(\varepsilon))$  must be moved farther away from the solutions of the Euler-Lagrange equations. Thus the magnitude of the first term on the right side of (16) integrated along  $\rho(AB(\varepsilon))$ ,  $\rho'(A'B(\varepsilon))$  increases as  $\varepsilon$  increases for each fixed variation  $\delta x$ . As above,  $O(\delta x(\varepsilon))$  increases with  $\varepsilon$ , implying a decrease in the intensity.

The above arguments also imply a symmetric intensity distribution as  $\varepsilon$  is varied over the interval zero to  $-1/2$ , and a repeat of the pattern as  $j$  is varied over the integers. Thus an interference pattern should be observed on the screen over a background of almost uniform but relatively low intensity as the major contributions have been estimated here.

Similar arguments may be used to estimate the variations in the intensity about peaks as  $j$  varies, resulting in a decrease in the intensity as  $|j|$  increases. This result is based on the fact that the term  $O(\delta^2)$  for each  $j$ , may be expressed as a sum of two terms, one being  $j$ -independent and the other, directly proportional to  $|j|$ .

Availability of two interfering beams originating at  $A$ ,  $A'$  and the equivalence of the physical conditions at these points have played a crucial role in the above analysis. As explained before, if one of the beams is blocked, the interference pattern is destroyed. Also, such a distribution should not be expected to result if the equivalence of  $A$  and  $A'$  is violated. This situation arises

when an attempt is made to observe the particle anywhere along the trajectory. Interaction with the detecting instrument changes the classical momentum of the particle say by  $\Delta P$ . It is straight forward to estimate the change  $\Delta S$  in the action which is very large for the macroscopic trajectories. This enables one to estimate  $\kappa$ . Consequently, a point B that was physical previously, either is no longer so or if physical, corresponds to a large value of  $|j|$ . In either case, the intensity transmitted to B by the interfering alternatives is negligible. Hence, the two beams transmit intensity as the classical beams of particles.

Above considerations indicate a wave-like behaviour of microscopic particles observed macroscopically as a collection while behaving as particles individually. This is in agreement with the observed behaviour [1, pp. 2-5]. These results obtained here from (13), are known to inspire the formalism of quantum mechanics.

### 4.3 The Aharonov-Bohm effect.

Additional insight into the behaviour of the particles as implied by the present extension may be gained by considering their response to a non-zero gauge field, as follows. The gauge transformation obtained by replacing  $\phi_\mu$  by  $\hat{\phi}_\mu$  will be denoted by  $\hat{U}_{BA}(\rho)$ . Let  $\{\rho\}$ ,  $\{\hat{\rho}\}$  be the collections of the solutions of (13), with  $\phi_\mu$ ,  $\hat{\phi}_\mu$  respectively. Assume that  $U_{BA}(\rho) \neq \hat{U}_{BA}(\rho)$  for a solution  $\rho(AB)$ . If  $U_{BA}(\rho)$  is replaced by  $\hat{U}_{BA}(\rho)$  in (13), then  $\rho(AB)$  is no longer a solution. The same conclusion holds for a path  $\hat{\rho}(A'B')$ . Thus, if the inequality holds for some of the solutions of (13) with  $\phi_\mu$ , or with  $\hat{\phi}_\mu$ , then the collections  $\{\rho\}$ , and  $\{\hat{\rho}\}$  of the physical paths are not identical. Therefore a change of potentials from  $\phi_\mu$  to  $\hat{\phi}_\mu$  should in general produce an observable effect. However, if  $U_{BA}(\rho') = \hat{U}_{BA}(\rho')$  for each  $\rho'(AB)$  in a collection  $\{\rho'\}$  large enough to include the union of  $\{\rho\}$  and  $\{\hat{\rho}\}$ , then (13) remains the same equation under the change from  $\phi_\mu$  to  $\hat{\phi}_\mu$ . Consequently, a change of potential from  $\phi_\mu$  to  $\hat{\phi}_\mu$  would not change the solutions  $\{\rho\}$ . Since the set of physical paths remains the same under this change, the response of the particles must remain unchanged also. Therefore, such a change of potentials will not alter the outcome of an experimental observation.

As an application, consider the Aharonov-Bohm effect [7]. In the corresponding experimental set up, the electrons travel in beams centered about paths  $\rho(ACB)$  and  $\rho'(ADB)$ , enclosing a non-zero magnetic field but shielded from it. Chambers used reflectors at  $C$  and  $D$  to obtain a configuration of piece-wise classical narrow beams centered about  $\rho(AC)$ ,  $\rho(CB)$ ,  $\rho'(AD)$  and  $\rho'(DB)$  [8]. The magnetic field was generated by placing a long coil carrying an electric current between the reflectors and perpendicular to the plane of the beams with one end in the plane. The electron beams were further shielded from the magnetic field. As the current in the coil is varied, the magnetic field varies accordingly. The classical Lagrangian for this system is the same as for the Lorentz equation.

As in the case of the double slit experiment, most of the electrons are transmitted by the interfering alternatives with parameter value increasing from  $A$  to  $B$  along  $\rho(ACB)$  and decreasing from  $B$  to  $A$  along  $\rho'(BDA)$ , taking value  $\tau_B$  at  $B$ . The estimates obtained in the treatment of the double slit experiment are valid for the present case as they were not restricted to a free particle. Some consideration should be given to the reflectors at  $C$  and  $D$ . Because of the continuity of the physical paths at points about  $A$ ,  $B$ ,  $C$ , and  $D$ ,  $\kappa(\ )$  cancels out. From Sec. 4.1, we have that most of the intensity transmitted along  $\rho(AC)$  reaches a small neighbourhood of  $C$  which remains almost within a macroscopically narrow beam. By the same argument, most of this intensity reaches a small neighbourhood of  $B$ . The same comment is

valid for  $\rho'(ADB)$ . The intensity along both of the beams is assumed equal. Consequently, the arguments of Sec. 4.2 can be used to conclude the existence of a similar interference pattern on the screen.

It follows from (13) that the interfering alternatives for an electro-magnetic potential  $\phi_\mu$  are the solutions of:

$$\exp \left[ i \oint (dS^P(x, \tau) - \phi_\mu dx^\mu) \right] \Phi_A = \Phi_A \quad (21)$$

where  $S^P(x, \tau)$  is the free particle part of the action and the integration is along the closed curves  $\rho_c(ACBDA)$ . Here the group element  $U_{BA}(\rho)$  is given by

$$U_{BA}(\rho) = \exp(i \int_{\rho(AB)} \phi_\mu dx^\mu)$$

It is clear that  $\phi_\mu$ -dependent part in (21) is  $U_{ABA}(\rho_c)$  which is given by  $U_{ABA}(\rho_c) = \exp(iF(\phi))$  where  $F(\phi)$  is the magnetic flux enclosed by  $\rho_c$ . As  $\rho_c$  is distorted,  $F(\phi)$  remains unchanged as long as the distorted closed path encloses the flux, which covers all of the paths of significance here as all of them surround the coil.

As  $F(\phi)$  varies to  $F(\hat{\phi})$ ,  $U_{ABA}(\rho_c) \neq \hat{U}_{ABA}(\rho_c)$  for any  $\rho_c$  unless

$$\begin{aligned} (F(\phi) - F(\hat{\phi})) &= \oint (\phi_\mu - \hat{\phi}_\mu) dx^\mu \\ &= 2\pi j \end{aligned} \quad (22)$$

with an arbitrary integer  $j$ . Whenever (22) is satisfied,  $U_{ABA}(\rho_c) = \hat{U}_{ABA}(\rho_c)$  for each curve  $\rho_c$  and hence the experimental observation with  $\hat{\phi}_\mu$  must be the same as with  $\phi_\mu$ . Thus the interference pattern on the screen should repeat itself periodically as the potential is varied continuously. The period is defined by (22).

Let  $\phi_\mu(\varepsilon)$  be a one parameter family of potentials with  $0 \leq \varepsilon \leq 1$ , such that  $(F(\phi(1)) - F(\phi(0))) = 2\pi$ , i.e.,  $\varepsilon$  covers one period. The intensity patterns corresponding to  $\phi_\mu(0)$  and  $\phi_\mu(1)$  are indistinguishable. Let the solutions of (21) with  $\phi_\mu$  replaced by  $\phi_\mu(\varepsilon)$  be  $\{\rho(\varepsilon)\}$ . Owing to the continuity of  $F(\phi(\varepsilon))$  with respect to  $\varepsilon$ ,  $\{\rho(\varepsilon)\}$  should vary continuously, implying a continuous variation of the corresponding interference pattern. As  $\varepsilon$  approaches one, the distribution of the intensity must return to the same as for  $\varepsilon = 0$ . Thus, each interference fringe should be expected to shift as  $\varepsilon$  varies from zero to one, from its position to the original location of the next.

Above conclusion agrees with the experimental observation [8,9]. It is pertinent to remark that the indistinguishability of  $\phi_\mu$  and  $\hat{\phi}_\mu$  that satisfy (22) is a direct consequence of (21) which is obtained from (13) and the fact that the physical paths in this case are closed in  $\mathcal{M}'$ . For this part of the conclusion, no estimates are needed.

The Aharonov-Bohm effect is an implication of the quantum mechanical equations [7] which were developed from different premises than the present formalism. Ingredients of the quantum mechanical deduction of this effect are the representation of the momenta  $p_\mu$  by  $-i\partial_\mu$  and the corresponding extension of the classical coupling scheme  $(p_\mu - \phi_\mu)$ . The former was inspired by the observed wave-like behaviour of particles and the later, in addition to being intuitive, sets  $\alpha = i$  in the London-Weyl [4,10] description of electro-magnetism. Here the major aspects of the Aharonov-Bohm effect are deduced directly from (13) without an appeal to any other theory.

Above considerations show that the wave-like behaviour of a particle in motion is a result of the properties of the physical paths. However, there is a crucial difference as described below.

Consider the double slit experiment. If the intensity pattern on the screen is a result of a wave motion, then there must be a point of zero intensity in between two bright regions. According to the present formulation, a point of minimum intensity exists but it can be seen that there must be some physical paths reaching every point on the screen, resulting in some intensity everywhere. If accurate enough determination of the intensity can be made, it may be possible to test whether the present theory or Quantum Mechanics provides a better description of motion. Nevertheless, major contribution to the intensity in the present formulation is the same as predicted by the wave motion. Thus one may use the results from the wave theory in building a theory of mechanics, at least approximately. While the above considerations justify use of the results from the theory of waves, it should be remarked that it is only for convenience rather than a physical attribute of the particles.

## 5 Equation of Motion

The classical action principle assigns a unique trajectory to a particle in motion between two points. The present extension (13), on the other hand, assigns many paths, but not all curves are allowed. Since it is impossible to assign a unique trajectory to a particle, as an alternative, one may describe its motion in terms of the intensity of the particles transmitted to a region in  $\mathcal{M}$  or  $\mathcal{M}'$  by the physical trajectories. This was done in Sec. 4 for a beam of free particles and for the double-slit experiment, but only approximately. Approximations were made in obtaining the estimates and by retaining only the major contributions. In a complete theory, all physical paths must be included and the contributions must be computed exactly. While such a theory is possible, it will require quite intricate computations for which a machinery is not yet developed. An approximate theory may be developed by exploiting the wave-like behaviour of the particles deduced in Sec. 4. In addition to simplifying the manipulations, this relates the present formulation with Quantum Mechanics which is instructive in itself.

Wave-like behaviour of particles and a possibility of describing their motion in terms of the probability densities associated with a collection of trajectories led Feynman to develop his path integral formulation of non-relativistic quantum mechanics [1,11]. The formalism was extended to the relativistic case in an analogous manner by introducing a proper time-like evolution parameter [12]. The wave-like behaviour of the particles was used to conclude that the intensity is the absolute square of the amplitude obtained by the law of superposition. The amplitude associated with a path  $\rho(AB)$  was taken to be proportional to  $\exp(iS_{BA}(\rho))$  which was based on a deduction by Dirac [13] of the behaviour of a quantum mechanical particle. Present formulation associates a phase-factor equal to  $\exp(iS_{BA}(\rho))$  with  $\rho(AB)$  whenever a classical description is possible in terms of a Lagrangian. The phases associated with a multiplicity of paths are shown in Sec. 4 to interfere in a manner that imparts wave-like properties to the particles in motion. A precise determination of a multitude of physical trajectories follows from (13). Thus all of the necessary assumptions required for the formulation of Feynman's postulates have been deduced from (13). It is straight forward to check that the assumption of particle following any out of all possible paths is extraneous to Feynman's postulates. Having yielded its basic assumptions, the gauge mechanical principle finds a natural expression within the framework of the path integral formalism. However, only the physical paths should be included in the computation of the total contribution.

*Postulate 1.* The probability of finding a particle in a region of space-time is the absolute

value of the sum of contributions from each physical path or its segment in the region.

*Postulate 2.* The contribution at a point  $C$  of a physical path  $\rho(AB)$  is equal to  $KV'_{CA}(\rho)\Phi_A$  where  $K$  is a path-independent constant.

Since the assumptions underlying the above postulates are deduced from (13), the formalism is self-consistent and based essentially on one assumption. Postulate 2. provides a mechanism for a computation of the total contribution from all trajectories by the techniques developed originally for the path-integral formulation. An equation of motion is developed below by this procedure and by isolating the contribution of the physical paths. Postulate 1. provides a means to obtain experimentally observable quantities from the solutions of the equation of motion.

Consider a point  $C$  on a physical path  $\rho(A'B')$ . Let  $\rho(AB)$  be the shortest segment of  $\rho(A'B')$  containing  $C$  such that  $A$  and  $B$  are equivalent to  $A'$  and  $B'$  respectively. Consider the pair of points  $A$  and  $A'$ . The pair  $B, B'$  is treated similarly. In view of the equivalence,  $V_{AA'}(\rho)\Phi_{A'}^P = \Phi_{A'}^P = \Phi_A^P$ , we have that  $V_{CA'}(\rho)\Phi_{A'}^P = V_{CA}(\rho)\Phi_A^P$ . Thus the contribution from  $\rho(A'C)$  is the same as that from  $\rho(AC)$ . Therefore it is sufficient to consider the minimal curves  $\rho(AB)$  instead of any larger physical paths containing  $\rho(AB)$ . As indicated in Sec. 3, interfering alternatives are included in this treatment.

The next step is to parameterize the minimal physical paths in a way that enables one to isolate their contribution. Since a single parameter is needed for all of the curves, standard parameterization by arc-length is inadequate. A suitable parameter was found in ref.[6] as follows. Let  $u'_\mu = \sum u_\mu$  where  $\sum$  denotes the sum over all paths of the type  $\rho(AB)$  with  $A$  being a variable point. For any such collection of curves, there is a Lorentz frame  $\mathcal{L}$  in which  $u'_\mu = 0$  for  $\mu = 1, 2, 3$ . A particle may thus be treated as being located at the origin of  $\mathcal{L}$ . Incidentally, the origin of  $\mathcal{L}$  coincides with the centre of mass of a fluid of uniform density and total mass  $m$  with an infinitesimal element flowing along each of  $\rho(AB)$  and with the arc-length in an appropriate Finsler space [3, Ch. 3.2]. Let  $z(\tau)$  be a parameterization of each path  $\rho(AB)$  with  $z(0) = A$ , where  $\tau$  is the proper time of  $\mathcal{L}$ . In  $\mathcal{L}$ , each of the curves  $\rho(AB)$  coincides with the straight line along  $\tau$ . Therefore,  $V_{CA}(\rho) = \exp(im\tau)$  and hence  $B = z(2\pi/m)$ . From Postulate 2, the contribution  $\psi'(x, \tau)$  at  $C = x$  is given by

$$\psi'(x, \tau) = \sum K'V'[x, z(\tau)]\Phi[z(0)] \quad (23)$$

where the sum is over all paths passing through  $x$  at  $\tau$ ;  $\Phi[z(0)] = \Phi_A$  and for each  $z(\tau)$ ,  $V'[x, z(\tau)] = V'_{CA}[z(\tau)]$ . The sum is the limit of a finite one with constant  $K'$  depending on the number of terms. Because of the continuity of the paths, the number of curves for  $\tau = 0$  is the same as for  $\tau = 2\pi/m$ . Also, for each physical path  $z(\tau)$ ,  $V[x, z(0)] = V[x, z(2\pi/m)] = 1$ , i.e.,  $V'[x, z(0)] = V'[x, z(2\pi/m)] = \kappa^{-1}(C)\kappa(A)$ . It follows that

$$\psi'(x, 0) = \psi'(x, 2\pi/m) \quad (24)$$

The boundary condition given by (24) provides a means to retain the contribution in (23) from the physical paths. Thus the proper time  $\tau$  of  $\mathcal{L}$  acquires a physical significance, which is treated below as an independent parameter as in [14]. The following derivation is essentially the same as in the standard path integral formulation.

Let  $[0, 2\pi/m]$  be divided into  $N$  equal intervals  $[\tau_j, \tau_{j+1}]$ ,  $j = 0, 1, \dots, N-1$ ; with  $\tau_0 = 0$ , and  $\tau_N = 2\pi/m$ . Consider all of the paths with  $z(\tau_k) = (x)_k$ . By the standard argument the function  $\psi'[(x)_k, \tau_k]$ , for each  $k$ , is given by

$$\begin{aligned}
\psi'[(x)_k, \tau_k] &= \int U^P[(x)_0, (x)_1] \cdots U^P[(x)_{k-1}, (x)_k] \\
&\quad \times U[(x)_k, (x)_{k-1}] \cdots U[(x)_1, (x)_0] \Phi[z(0)] \\
&\quad \times \frac{d(x)_0}{Q} \cdots \frac{d(x)_{k-1}}{Q}
\end{aligned} \tag{25}$$

where  $U^P[(x)_{j+1}, (x)_j] = \{U^P[(x)_j, (x)_{j+1}]\}^{-1} = U_{B'A'}^P[z(\tau)]$ ,  $U[(x)_{j+1}, (x)_j] = U_{B'A'}[z(\tau)]$  with  $A' = (x)_j$ ,  $B' = (x)_{j+1}$ , and  $Q$  is a normalization constant. Set  $(x)_k = y$ ,  $\tau_k = \tau$ ,  $(x)_{k+1} = x$  and  $\tau_{k+1} = \tau_k + \epsilon$ . It follows from (25) that

$$\psi'(x, \tau + \epsilon) = \frac{1}{Q} \int U^P(y, x) U(x, y) \psi'(y, \tau) dy \tag{26}$$

A curve  $z(\tau)$  in  $\mathcal{M}$  may be arbitrarily closely approximated by  $z_N(\tau)$  for large enough  $N$ , where  $z_N(\tau_j) = z(\tau_j)$ ,  $j = 0, 1, \dots, N$ ; and in each of the intervals  $[\tau_j, \tau_{j+1}]$ ,  $z_N(\tau)$  is the geodesic line. The element  $U^P(y, x) = U_{y,x}^P[z(\tau)]$  may be approximated by

$$U_{y,x}^P[z_N(\tau)] = \exp \left[ iS^P(x, y) \right]$$

where  $S^P(x, y)$  is Hamilton's principal function for a 'free' particle of mass  $m$  from  $x$  to a variable point  $y$ . Here the Lagrangian is  $L^P$  with  $\tau$  being the proper time of  $\mathcal{L}$ . The action is given by

$$S^P(x, y) = -\frac{m}{2\epsilon} g_{\mu\nu} \xi^\mu \xi^\nu - \frac{m}{2} \epsilon$$

where  $\xi^\mu = (x^\mu - y^\mu)$ . Also,  $U(x, y)$  is approximated by  $U_{x,y}[z_N(\tau)]$  up to the desired order which is given by

$$\begin{aligned}
U_{x,y}[z_N(\tau)] &= 1 + i\phi_\mu(x) \xi^\mu - \frac{1}{2} [i\phi_{\mu,\nu} + \phi_\mu \phi_\nu] \xi^\mu \xi^\nu \\
&\quad + \text{higher order terms.}
\end{aligned}$$

Let  $\psi(x, \tau) = \exp(im\tau/2) \psi'(x, \tau)$ , then it follows from (24) that

$$\psi(x, 0) = -\psi(x, 2\pi/m), \tag{27}$$

With the above substitutions, from (26), we have

$$\psi(x, \tau + \epsilon) = \frac{1}{Q} \int \exp \left[ -\frac{im}{2\epsilon} g_{\mu\nu} \xi^\mu \xi^\nu \right] U_{x,y}[z_N(\tau)] \psi(x - \xi, \tau) d\xi \tag{28}$$

Eq. (28) holds exactly in the limit of infinite  $N$ , equivalently  $\epsilon = 0$ . As such it holds up to the first order in  $\epsilon$ , which is sufficient for the present.

Expanding  $\psi(x, \tau + \epsilon)$  and  $\psi(x - \xi, \tau)$  in a Taylor series about the point  $(x, \tau)$  and comparing the coefficients of  $\epsilon^j$ ,  $j = 0, 1$ , yields  $Q = -i(2\pi\epsilon/m)^2$  and

$$i \frac{\partial \psi}{\partial \tau} = -\frac{1}{2m} \Pi_\mu \Pi^\mu \psi \tag{29}$$

where  $\Pi_\mu = (i\partial/\partial x^\mu \cdot 1 + \phi_\mu)$ . In view of the boundary condition (27),  $\psi$  may be expressed as

$$\psi(x, \tau) = \sum_{-\infty}^{\infty} \psi_n(x) \omega_n(\tau)$$

where for each  $n$ ,  $\omega_n(\tau) = \sqrt{m/2\pi} \exp[i(n + 1/2)m\tau]$  and  $\psi_n$  satisfies

$$\begin{aligned} \Pi_\mu \Pi^\mu \psi_n &= (2n + 1)m^2 \psi_n \\ n &= 0, \pm 1, \pm 2, \dots \end{aligned} \quad (30)$$

For  $n = 0$ , (30) reduces to the Klein-Gordon equation in an electro-magnetic field.

Equation of motion (29) termed the generalized Schrödinger equation, was first conjectured by Stückelberg [15]. The boundary condition (27) is a direct result of the definition of the physical paths provided by (13). As shown above, this boundary condition is crucial in relating (29) to the Klein-Gordon equation. If all trajectories are allowed to contribute, the resulting equation is still (29) but without the boundary condition (27). Feynman [12] used this equation to deduce the Klein-Gordon equation by restricting the solution to the form  $\psi_0(x)\omega_0(\tau)$ . Present treatment relates (29) with the Klein-Gordon equation (30) quite naturally. Further to the arguments of Sec. 4, this result provides additional support for the assumption (13).

## 6 Further Developments

The above procedure has also been used to develop an equation of motion in a Riemannian space where the resulting theory is conceptually clearer [16]. In particular, the arc-length serves as an appropriate evolution parameter which also indicates that a more accurate theory would be easier to develop in the setting of a Riemannian space. For the present, the analogue of the generalized Schrödinger equation in a Riemannian space reads as

$$2im' \frac{\partial \psi}{\partial \tau} = [\partial_\mu \partial^\mu + \frac{1}{3}R]\psi \quad (31)$$

where  $\partial_\mu$  are the components of the covariant derivative,  $R$  is the curvature scalar and  $\mu$  runs over the dimension of the space. The parameter  $m'$  is determined by the classical Hamilton's equations. For a gravitational field  $m' = m$ . The boundary condition  $\psi(x, 2\pi/m') = -\psi(x, 0)$  still holds which reduces (31) into infinitely many equations:

$$-\partial_\mu \partial^\mu \psi_n = [(2n + 1)m^2 + \frac{1}{3}R]\psi_n, \quad n = 0, \pm 1, \pm 2, \dots \quad (32)$$

For  $n=0$ , in standard units (32) reads as

$$-\hbar^2 \partial_\mu \partial^\mu \psi_0 = [m^2 c^4 + \frac{1}{3}\hbar^2 R]\psi_0 \quad (33)$$

where  $c$  is the speed of light,  $\hbar = h/2\pi$  and  $h$  is Planck's constant.

Motion of a charged particle in an electro-magnetic field may be described in the setting of a Riemannian space in the Kaluza-Klein framework [17]. The equations of motion may be obtained as special cases of the equations in the Riemannian spaces or independently [18]. The resulting generalized Schrödinger equation is given by

$$\frac{\partial \psi}{\partial \tau} = \frac{1}{2im'} \left[ \left( \frac{\partial}{\partial x^\mu} - \phi_\mu \frac{\partial}{\partial x^5} \right) \left( \frac{\partial}{\partial x_\mu} - \phi^\mu \frac{\partial}{\partial x^5} \right) - \frac{1}{12} F_{\mu\nu} F^{\mu\nu} - \left( \frac{\partial}{\partial x^5} \right)^2 \right] \psi \quad (34)$$



where  $m' = \sqrt{m^2 - e^2}$  with  $e$  being the charge. In view of the compactness of the fifth dimension and the associated periodicity,  $\psi$  may be expanded in a Fourier series:

$$\psi = \sum_{k=-\infty}^{\infty} \psi_k(x, \tau) \exp[iekx^5]$$

where  $e = p_5$ , and since it will cause no confusion,  $x$  now denotes a point in the Minkowski manifold instead of the Kaluza-Klein. Substitution of the expansion for  $\psi$  in (34) decomposes it into a set of generalized Schrödinger type equations with charge quantized in units of  $e$ :

$$\begin{aligned} -2im' \frac{\partial \psi_k}{\partial \tau} &= \left[ \left( i \frac{\partial}{\partial x^\mu} + ek\phi_\mu \right) \left( i \frac{\partial}{\partial x_\mu} + ek\phi^\mu \right) + \frac{1}{12} F_{\mu\nu} F^{\mu\nu} - (ek)^2 \right] \psi_k \\ k &= 0, \pm 1, \pm 2, \dots \end{aligned} \quad (35)$$

Further, in view of the boundary condition  $\psi(x, 2\pi/m') = -\psi(x, 0)$  with respect to  $\tau$ ,  $\psi_k$  may be expressed as

$$\psi_k(x, \tau) = \sum_{n=-\infty}^{\infty} \psi_{kn}(x) \exp[i(n + 1/2)m'\tau]$$

reducing (35) to

$$\begin{aligned} \left( i \frac{\partial}{\partial x^\mu} + ek\phi_\mu \right) \left( i \frac{\partial}{\partial x_\mu} + ek\phi^\mu \right) \psi_{kn} &= \\ [(2n + 1)(m^2 - e^2) + e^2 k^2 - \frac{1}{12} F_{\mu\nu} F^{\mu\nu}] \psi_{kn} & \\ k, n &= 0, \pm 1, \pm 2, \dots \end{aligned} \quad (36)$$

For  $n = 0$  and  $k = 1$ , (36) is the Klein-Gordon equation with  $m^2$  modified by  $(-F_{\mu\nu} F^{\mu\nu}/12)$ , one third of the curvature scalar of the five dimensional Kaluza-Klein space. In standard units, the equation for  $\psi_{10}$  is expressed as

$$\left( i\hbar \frac{\partial}{\partial x^\mu} + \frac{e}{c} \phi_\mu \right) \left( i\hbar \frac{\partial}{\partial x_\mu} + \frac{e}{c} \phi^\mu \right) \psi_{10} = [m^2 c^4 - \frac{1}{6} G \hbar^2 F_{\mu\nu} F^{\mu\nu}] \psi_{10} \quad (37)$$

where  $G = 6.66 \times 10^{-8} \text{dyn.cm}^2/\text{gm}^2$  is the universal gravitational constant.

The above methods are applicable also to the case of a general gauge field in the setting of the Minkowski manifold [5] or a Riemannian space in the Kaluza-Klein framework, and to the treatment of the spinors [19].

The next major step in constructing a complete theory of mechanics in the present framework would be to abandon the path-integral formalism and compute the intensity transmitted by the physical paths directly by solving the functional equations. Comparisons with other theories e.g. Bohmian mechanics is desirable. The studies of other phenomena e.g. tunneling and behaviour of the correlated particles, even with the level of accuracy of Sec. 4, should prove instructive. Also, the physical implications of the additional equations arising here should be investigated.

## 7 Concluding Remarks

The action principle determines a particle trajectory by requiring the action to be stationary under all small deformations. In group theoretical terms, this results in a requirement of equivalence between the elements associated with a subset of the closed curves up to the first order only. In this article, the classical action principle is extended to require the equivalence of the global elements associated with all of the curves. Solutions of the resulting equation form an infinite subset, termed the physical paths, to assign to a particle in motion.

Properties of the physical paths impart wave-like properties to a particle in motion. The wave-like behaviour of particles and the multiplicity of allowed paths form the basis of the path integral formulation. An imaginary value of  $\alpha$  yielded by the present extension, implies the compactness of the gauge groups which is inherent in quantum mechanical equations in gauge fields. Consequent description of the influence of the field enclosed by a closed curve on the particles, as is the case with the Aharonov-Bohm effect, is described by (13) to a large extent without an appeal to any other theory. Thus the present formulation develops a coherent theory unifying various treatments underlying the existing quantum mechanics without involving its usual assumptions.

The above results lead naturally to Feynman's path integral formalism with physical paths being the contributing members. The criterion imposed by (13) on the physical paths plays a crucial role in the deduction of the above results, some of which have been used to justify the use of the path integral formalism. Thus the present formulation is self-consistent.

In the present paper we have used a proper time-like parameter to convert the problem of isolating the contribution from the physical paths into a boundary condition on (29). This type of parameter was introduced in a rather *ad hoc* manner by several authors [12,14,15]. Here this parameter gains a clearer physical significance. A need for a five-dimensional relativistic wave equation has been felt for a long time, for the existing equations suffer from some conceptual difficulties. In response to this need, Stückelberg originally conjectured the generalized Schrödinger equation for a particle in an Abelian gauge field [15]. There is a renewed interest in this equation to interpret it in a more satisfactory framework than a conjecture, as well as to study its implications (see e.g., [20]). Present formalism provides a systematic derivation of the generalized Schrödinger equation.

In addition to accepting the conjecture of Stückelberg, Feynman selected a particular set of periodic solutions to deduce the Klein-Gordon equation from the generalized Schrödinger equation. As pointed out above, the physical paths are characterized by a boundary condition on (29). This boundary condition confines the solution to a set described by a class of periodic functions. As a consequence, the equation decomposes into countably many four dimensional equations, one of them being the Klein-Gordon equation. Thus the resulting boundary condition provides an additional justification for the present treatment.

Classical description of motion is quite accurate at the macroscopic scale. Quantum Mechanics modifies these results only slightly but conceptually it is fundamentally different. It also appeals to experimental observations for its underlying assumptions without offering conceptual clarity. The present formulation extends Classical Mechanics yielding these assumptions and various conjectures in a coherent framework. Thus the gauge mechanical principle offers a more satisfactory basis for the formulation of mechanics. In particular, it eliminates the need for a direct assumption of wave nature of a particle in motion which underlies the well known difficulties with Quantum Mechanics. It is pertinent to remark that while the present theory associates a somewhat objective meaning to a particle in motion, an element of randomness

remains in the availability of the equally likely, infinitely many paths.

Quantum Mechanics results as an approximation to the present theory, presumably quite accurate. Deviations from Quantum Mechanics are pointed out, and directions for further investigations, and to construct a more accurate and complete theory, are indicated.

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### References

- [1] 1. R.P. Feynman and A.R. Hibbs, Quantum Mechanics and Path Integrals, McGraw-Hill, New York, 1965, pp. 2-9.
- [2] 2. J. Horgan, Quantum Philosophy, Scientific American, July 1992, pp. 94-104.
- [3] 3. H. Rund, The Hamilton-Jacobi Theory in the Calculus of Variations, Van Nostrand, London, 1966.
- [4] 4. H. Weyl, Ann. Phys. (Leipzig) 59 (1919) 101; H. Weyl, Space Time Matter (Translated by H.L. Brose) Dover, New York, 1951, Ch. IV, Sec. 35.
- [5] 5. S.R. Vatsya, Can. J. Phys., 73 (1995) 85.
- [6] 6. S.R. Vatsya, Can. J. Phys., 67, (1989) 634.
- [7] 7. Y. Aharonov and D. Bohm, Phys. Rev. 115 (1959) 485.
- [8] 8. R.G. Chambers, Phys. Rev. Lett. 5 (1959) 3.
- [9] 9. A. Tonomura, N. Osakabe, T. Matsuda, T. Kawasaki and J. Endo, Phys. Rev. Lett. 56 (1986) 792.
- [10] 10. F. London, Z. Phys. 42 (1927) 375.
- [11] 11. R.P. Feynman, Rev. Mod. Phys. 20 (1948) 367.
- [12] 12. R.P. Feynman, Phys. Rev. 80 (1950) 440.
- [13] 13. P.A.M. Dirac, The Principles of Quantum Mechanics (Fourth edition) The Clarendon Press, Oxford, England (1958) Sec. 32.
- [14] 14. V. Fock, Phys. Zeit. Sowjet. un. 12 (1937) 404.
- [15] 15. E.C.G. Stückelberg, Helv. Phys. Acta. 14 (1941) 322; Ibid. 15 (1942) 23.
- [16] 16. S.R. Vatsya, Mechanics of a particle in a Riemannian manifold, Unpublished.
- [17] 17. Th. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Berlin, Math. Phys. K1 (1921) 966; O. Klein, Z. Phys. 37 (1926) 895.
- [18] 18. S.R. Vatsya, Mechanics of a charged particle on the Kaluza-Klein background, Can. J. Phys. 73 (1995).
- [19] 19. S.R. Vatsya, Work in progress.
- [20] 20. A. Kyprianidis, Phys. Rep. 155 (1987) 1.